5.4, 5.5 Definite integrals

Recall $\int f(x)\,dx = F(x) + \underline{\underline{C}}$ where $F'(x) = f(x)$: indefinite

Here we look at: $\int_a^b f(x)\,dx$ where $a \le x \le b$, $f(x)$ is continuous over (a,b)

Definite integral

Fundamental Theorem of Calculus - I

Suppose $f(x)$ is continuous over the interval (a,b), then $\int_a^b f(x)dx = F(b) - F(a)$

where $F(x)$ is the antiderivative of $f(x)$

Ex. Evaluate $\int_0^1 x\,dx$

Solution:

Let $f(x) = x$, $F(x) = \int f(x)\,dx = \int x\,dx$

$= \frac{1}{2}x^2 + C$

- Now when $x = 1$ (upper bound), we have $F(1) = \frac{1}{2}(1) + C$
$= \frac{1}{2} + C$
- When $x = 0$ (lower bound), we have $F(0) = \frac{1}{2}(0) + C$

So $\int_0^1 x\,dx = F(1) - F(0)$

$= \frac{1}{2} + C - (0 + C) = \frac{1}{2} + C - C = \boxed{\frac{1}{2}}$

$= \boxed{\frac{1}{2}}$

Example $\int_2^9 \left(\frac{8}{x^2} - 1\right) dx$ * ENTER $y = \frac{8}{x^2} - 1$ on your graphing utility, then do: 2ND TRACE 7

lower endpoint "2" then upper endpoint "9"

ENTER : -3.8888889

*(By hand) $\int_2^9 \left(\frac{8}{x^2} - 1\right) dx$

First find $F(x) = \int \left(\frac{8}{x^2} - 1\right) dx. = \int (8x^{-2} - 1)\, dx$

$$= \frac{8x^{-1}}{-1} - x = \frac{-8}{x} - x$$

So $\int_2^9 \left(\frac{8}{x^2} - 1\right) dx = \left[\underbrace{\frac{-8}{x} - x}_{F(x)} \right]_2^9 = \underbrace{\frac{-8}{9} - 9}_{F(9)} - \underbrace{\left(\frac{-8}{2} - 2\right)}_{F(2)}$

$$= \frac{-8}{9} - 3 = \boxed{\frac{-35}{9}}$$

Verify that $\frac{-35}{9} \simeq -3.8888889$

Ex: $\displaystyle\int_0^{\pi} (6 + \sin x)\,dx$

$$= \Big[6x - \cos x\Big]_0^{\pi} = 6(\pi) - \cos(\pi) - 6(0) + \cos(0)$$

$$= 6\pi - (-1) - 0 + 1$$

$$= 2 + 6\pi$$

Ex: $\displaystyle\int_0^{4} (2^x + 5)\,dx$

$$= \left[\frac{1}{\ln 2} 2^x + 5x\right]_0^4$$

Recall: $(2^u)' = u' 2^u \cdot \ln 2$

$(2^x)' = 2^x \ln 2$

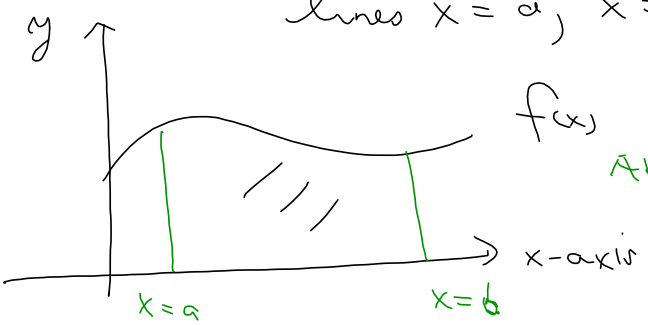
$\left(\frac{1}{\ln 2} 2^x\right)' = \frac{2^x}{\ln 2} \cdot \ln 2 = 2^x$

$$= \frac{2^4}{\ln 2} + 20 - \frac{2^0}{\ln 2} - 0$$

$$= \frac{16}{\ln 2} - \frac{1}{\ln 2} + 20 = \frac{15}{\ln 2} + 20$$

Application:

$\displaystyle\int_a^b f(x)\,dx$ is the area of the region above x-axis (or below x-axis) that is bounded by $f(x)$, and the vertical lines $x = a$, $x = b$



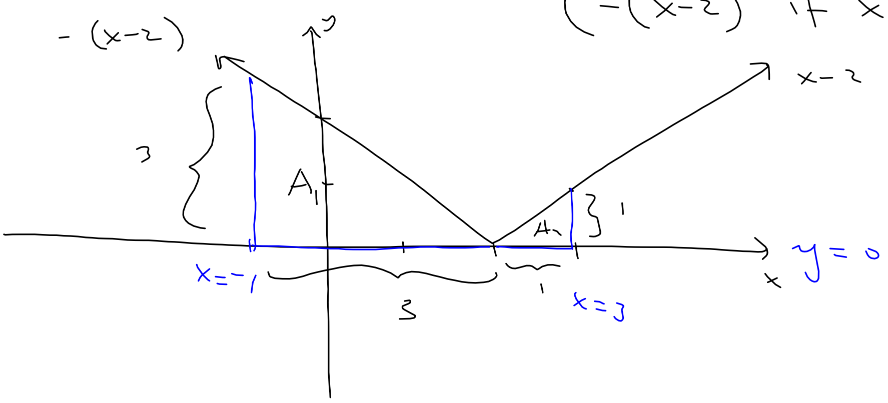
Area $= \displaystyle\int_a^b f(x)\,dx$

Ex: Find the area of a region bounded by $y=0$ (x-axis), $x=-1$, $x=3$ and

$$f(x) = |x-2|$$

Solution:

$$f(x) = |x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$



One way: geometry:

$$A = A_1 + A_2$$

$$= \frac{9}{2} + \frac{1}{2} = \boxed{5 \text{ unit}^2}$$

Other way: Calculus

$$\int_{-1}^{3} |x-2|\,dx = \int_{-1}^{2} -(x-2)\,dx + \int_{2}^{3} (x-2)\,dx$$

$$= \left[-\frac{1}{2}x^2 + 2x\right]_{-1}^{2} + \left[\frac{1}{2}x^2 - 2x\right]_{2}^{3}$$

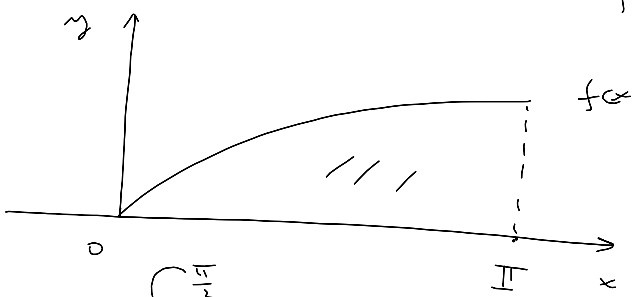
$$= \left[-2+4-\left(-\frac{1}{2}-2\right)\right] + \left[\frac{9}{2}-6-(2-4)\right]$$

$$= \left[2+\frac{5}{2}\right] + \left[\frac{9}{2}-4\right]$$

$$= \frac{9}{2} + \frac{1}{2} = \frac{10}{2} = \boxed{5 \text{ unit}^2}$$

$$A_1 + A_2 = A$$

#21 Determine the area of the region

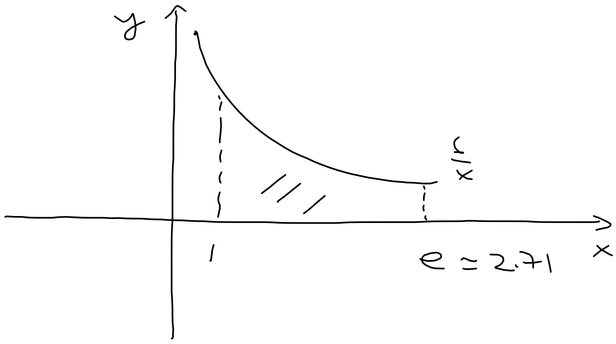


$$A = \int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= \left[-\cos\left(\tfrac{\pi}{2}\right) - \left(-\cos(0)\right)\right] = \left[0 - (-1)\right]$$

$$\boxed{A = 1 \text{ unit}^2}$$

#25 Find the area of the region bounded by $y = \frac{6}{x}$, $x = 1$, $x = e$, $y = 0$ (x-axis)



$$\text{Area} = \int_1^e \frac{6}{x}\,dx$$

$$= 6\int_1^e \frac{1}{x}\,dx$$

$$= 6\Big[\ln|x|\Big]_1^e$$

$$= 6\left[\ln e - \ln 1\right]$$

$$= 6\left[1 - 0\right]$$

$$\boxed{A = 6\,\text{unit}^2}$$

Fundamental theorem of calculus part II

Suppose $f(x)$ is continuous over an open interval I. Let $x \in I$, then

$$A(x) = \int_a^x f(t)\,dt \quad \text{and}$$

$$A'(x) = f(x)$$

In general, if $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then

$$F'(x) = \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\Big[F(t)\Big]_{g(x)}^{h(x)}$$

$$= \frac{d}{dx}\Big[F(h(x)) - F(g(x))\Big]$$

$$\boxed{F'(x) = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

Ex: #27 $F(x)=\int_{-3}^{x}\left(\sqrt{t^4+7}\right)dt$, find $F'(x)$

$$\sqrt{x^4+7}\,(1) - \cancel{\left(\sqrt{(-3)^4+7}\right)}(0) \rightarrow 0$$

$$= \sqrt{x^4+7}$$

30 $F(x) = \int_{x}^{x^5} \frac{\sin t}{1-\ln t}\,dt, \quad x>0$

find $F'(x)$

Ans:

$$\left[\frac{\sin(x^5)}{1-\ln x^5}(5x^4) - \frac{\sin x}{1-\ln x}\cdot(1)\right]$$

31 (modified) $F(x) = \int_{\sqrt{x}}^{\frac{1}{x}} (\tan t^2)\,dt$

$$\tan\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) - (\tan x)\cdot\frac{1}{2\sqrt{x}}$$